

# Complex almost contact metric structures on complex hypersurfaces in hyperkähler manifolds

MITSUHIRO IMADA

National Institute of Technology, Ibaraki Collage

November 4, 2015

Abstract: In this paper, we construct complex almost contact metric structures on complex hypersurfaces in hyperkähler manifolds. This construction is analogous to that in contact geometry.

## 1 Introduction

The theory of complex contact geometry started with the papers of Kobayashi [5], as a variant of real contact geometry. More recent examples, including complex projective space and the complex Heisenberg group, are given in [1]. Ishihara and Konishi [4] defined the so-called I-K normality of complex contact manifolds as for Sasakian manifolds in real contact geometry. In this paper, we construct complex almost contact manifolds from hyperkähler manifolds. Leaving the detailed notion of hyperkähler manifolds to Definition 3.1, we state the first main result as follows:

**Theorem A (Theorem 3.3.)** Let  $(\widetilde{M}, J_1, J_2, J_3, \tilde{g})$  be a hyperkähler manifold and  $M$  be a complex hypersurface of  $\widetilde{M}$ . The inclusion  $\iota : M \longrightarrow \widetilde{M}$  canonically induces a complex almost contact metric structure on  $M$ .

This main result is an analogous to Morimoto [7]. He shows that real hypersurfaces in Kähler manifolds equip an almost contact metric structure induced from the ambient Kähler structure.

In addition, we show that covariant derivatives of tensors belonging to complex almost contact metric structures have the following forms.

**Theorem B (Theorem 4.1.)** Let  $(G, H, J, u, v, U, V, g)$  be a complex almost

---

Keywords: Complex contact manifolds; ; 53D10

contact metric structure on a complex hypersurface  $M$ . Then the derivatives of  $G$  and  $H$  have the following forms:

$$\begin{aligned}(\nabla_X G)Y &= -u(Y)AX + v(Y)JAX + g(AX, Y)U - g(JAX, Y)V, \\(\nabla_X H)Y &= -u(Y)JAX - v(Y)AX + g(AX, Y)V + g(JAX, Y)U.\end{aligned}$$

This paper is organized as follows. In section 2, we recall definitions of complex contact manifolds and hyperkähler manifolds. In section 3, we prove the main theorem which constructs complex almost contact metric structures on complex hypersurfaces in hyperkähler manifolds. In section 4, we give some results of tensor calculations of these complex almost contact metric structures.

## 2 Definitions

We first recall the notion of complex contact metric manifolds [1].

**Definition 2.1.** Let  $M$  be a complex manifold with  $\dim_{\mathbb{C}} M = 2n + 1$  and  $J$  the complex structure on  $M$ .  $M$  is called a complex contact manifold if there exists an open covering  $\mathcal{U} = \{\mathcal{O}_\lambda\}$  of  $M$  such that:

- 1) On each  $\mathcal{O}_\lambda$  there is a holomorphic 1-form  $\omega_\lambda$  with  $\omega_\lambda \wedge (d\omega_\lambda)^n \neq 0$  everywhere;
- 2) If  $\mathcal{O}_\lambda \cap \mathcal{O}_\mu \neq \emptyset$ , there is a nonvanishing holomorphic function  $h_{\lambda\mu}$  on  $\mathcal{O}_\lambda \cap \mathcal{O}_\mu$  such that

$$(1) \quad \omega_\lambda = h_{\lambda\mu} \omega_\mu \quad \text{in } \mathcal{O}_\lambda \cap \mathcal{O}_\mu.$$

For each  $\mathcal{O}_\lambda$ , we define a distribution  $\mathcal{H}_\lambda = \{X \in T\mathcal{O}_\lambda \mid \omega_\lambda(X) = 0\}$ . Note that the  $h_{\lambda\mu}$  are nonvanishing, and  $\mathcal{H}_\lambda = \mathcal{H}_\mu$  on  $\mathcal{O}_\lambda \cap \mathcal{O}_\mu$ . Thus  $\mathcal{H} = \cup \mathcal{H}_\lambda$  is a holomorphic, nonintegrable subbundle on  $M$ , called the horizontal subbundle.

**Definition 2.2.** Let  $M$  be a complex manifold with  $\dim_{\mathbb{C}} = 2n + 1$  and  $J$  a complex structure. Let  $g$  be a Hermitian metric.  $M$  is called a complex almost contact metric manifold if there exists an open covering  $\mathcal{U} = \{\mathcal{O}_\lambda\}$  of  $M$  such that:

- 1) On each  $\mathcal{O}_\lambda$  there are 1-forms  $u_\lambda$  and  $v_\lambda = -u_\lambda J$ , (1,1) tensors  $G_\lambda$  and  $H_\lambda = JG_\lambda$ , unit vector fields  $U_\lambda$  and  $V_\lambda = JU_\lambda$  such that

$$\begin{aligned}(2) \quad &G_\lambda J_\lambda = -J_\lambda G_\lambda, \quad H_\lambda^2 = G_\lambda^2 = -id + u_\lambda \otimes U_\lambda + v_\lambda \otimes V_\lambda, \\&g(G_\lambda X, Y) = -g(X, G_\lambda Y), \quad g(U_\lambda, X) = u_\lambda(X), \\&G_\lambda U_\lambda = 0, \quad u_\lambda(U_\lambda) = 1;\end{aligned}$$

- 2) If  $\mathcal{O}_\lambda \cap \mathcal{O}_\mu \neq \emptyset$ , there are functions  $a, b$  on  $\mathcal{O}_\lambda \cap \mathcal{O}_\mu$  such that

$$\begin{aligned}
(3) \quad & u_\mu = au_\lambda - bv_\lambda, \quad v_\mu = bu_\lambda + av_\lambda, \\
& G_\mu = aG_\lambda - bH_\lambda, \quad H_\mu = bG_\lambda + aH_\lambda, \\
& a^2 + b^2 = 1.
\end{aligned}$$

**Definition 2.3.** Let  $(M, \{\omega_\lambda\})$  be a complex contact manifold with complex contact structure  $J$  and Hermitian metric  $g$ . We call  $(M, J, G, u, U, g)$  a complex contact metric manifold if there exists an open covering  $\mathcal{U} = \{\mathcal{O}_\lambda\}$  of  $M$  such that (here and below  $G = G_\lambda$ , etc) :

- 1) On each  $\mathcal{O}_\lambda$  there is a local  $(1,1)$  tensor  $G_\lambda$  such that  $(u_\lambda, v_\lambda, U_\lambda, V_\lambda, G_\lambda, H_\lambda = G_\lambda J, g)$  is an almost contact metric structure on  $M$ ;
- 2)  $g(X, G_\lambda Y) = du_\lambda(X, Y) + (\sigma_\lambda \wedge v_\lambda)(X, Y)$  and  $g(X, H_\lambda Y) = dv_\lambda(X, Y) - (\sigma_\lambda \wedge u_\lambda)(X, Y)$ , where  $\sigma_\lambda(X) = g(\nabla_X U_\lambda, V_\lambda)$  with  $\nabla$  the Levi-Civita connection with respect to  $g$ .

**Remark 2.4.** Foreman [?] showed the existence of complex contact metric structures on complex contact manifolds.

**Remark 2.5.** We can locally choose orthonormal vectors  $X_1, \dots, X_n$  in  $\mathcal{H}$  such that  $\{X_i, JX_i, GX_i, HX_i, U, V \mid 1 \leq i \leq n\}$  is an orthonormal basis of the tangent spaces of  $U_\alpha$ .

We recall the definition of I-K normality introduced by Ishihara and Konishi [3] for (almost) complex contact metric structures. We set the two tensor fields  $S$  and  $T$  by,

$$\begin{aligned}
(4) \quad S(X, Y) &= [G, G](X, Y) + 2g(X, GY)U - 2g(X, HY)V \\
&\quad + 2v(Y)HX - 2v(X)HY + \sigma(GY)HX \\
&\quad - \sigma(GX)HY + \sigma(X)GHY - \sigma(Y)GHX, \\
(5) \quad T(X, Y) &= [H, H](X, Y) - 2g(X, GY)U + 2g(X, HY)V \\
&\quad + 2u(Y)GX - 2u(X)GY + \sigma(HX)GY \\
&\quad - \sigma(HY)GX + \sigma(X)GHY - \sigma(Y)GHX.
\end{aligned}$$

**Definition 2.6.** A complex contact manifold  $M$  is I-K normal if the tensors  $S$  and  $T$  both vanish.

**Remark 2.7.** I-K normality implies that the underlying Hermitian manifold  $(M, J, g)$  is a Kähler manifold (cf. [4]).

### 3 Constructions

In this chapter, we construct a complex almost contact metric structure on complex hypersurfaces in hyperkähler manifolds. At first, we recall the definition of hyperkähler manifolds.

**Definition 3.1.**  $(M, J_1, J_2, J_3, g)$  is a hyperkähler manifold if  $J_1, J_2, J_3$  are complex structures on a complex manifold  $M$  satisfying

$$J_1^2 = J_2^2 = J_3^2 = J_1 J_2 J_3 = -id,$$

**Definition 3.2.** A complex contact manifold  $M$  is I-K normal if the tensors  $S$  and  $T$  both vanish.

Let  $(\tilde{M}, J_1, J_2, J_3, \tilde{g})$  be a hyperkähler manifold and  $M$  be a complex hypersurface in  $\tilde{M}$ . For each  $\tilde{X} \in T\tilde{M}$ , we decompose  $\tilde{X}$  as follows:

$$(6) \quad \tilde{X} = X + \tilde{g}(\tilde{X}, \xi)\xi + \tilde{g}(\tilde{X}, J_1\xi)J_1\xi$$

where  $X$  is the component of  $\tilde{X}$  tangent to  $M$ , and  $\xi$  and  $J_1\xi$  are normal to  $M$ . Applying  $J_1$  to (6), we have

$$J_1\tilde{X} = J_1X + \tilde{g}(\tilde{X}, \xi)J_1\xi - \tilde{g}(\tilde{X}, J_1\xi)\xi.$$

Applying  $J_2$  to (6), we have

$$J_2\tilde{X} = J_2X + \tilde{g}(\tilde{X}, \xi)J_2\xi - \tilde{g}(\tilde{X}, J_1\xi)J_3\xi.$$

Since  $J_2\xi$  is not tangent to  $TM$ , we decompose  $J_2X$  as follows:

$$(7) \quad J_2X = GX + \tilde{g}(J_2X, \xi)\xi + \tilde{g}(J_2X, J_1\xi)J_1\xi,$$

where  $GX$  is the component of  $J_2X$  tangent to  $TM$ . Then

$$\begin{aligned} J_2\tilde{X} &= GX + \tilde{g}(J_2X, \xi)\xi + \tilde{g}(J_2X, J_1\xi)J_1\xi + \tilde{g}(\tilde{X}, \xi)J_2\xi - \tilde{g}(\tilde{X}, J_1\xi)J_3\xi \\ &= GX + \tilde{g}(\tilde{X}, \xi)J_2\xi - \tilde{g}(\tilde{X}, J_1\xi)J_3\xi - \tilde{g}(X, J_2\xi)\xi + \tilde{g}(X, J_3\xi)J_1\xi \\ (8) \quad &= GX + \tilde{g}(\tilde{X}, \xi)J_2\xi - \tilde{g}(\tilde{X}, J_1\xi)J_3\xi - g(X, J_2\xi)\xi + g(X, J_3\xi)J_1\xi. \end{aligned}$$

Again applying  $J_2$  to (8), we have

$$\begin{aligned} J_2^2\tilde{X} &= J_2GX - \tilde{g}(\tilde{X}, \xi)\xi - \tilde{g}(\tilde{X}, J_1\xi)J_1\xi \\ &\quad - g(X, J_2\xi)J_2\xi - g(X, J_3\xi)J_3\xi \\ &= G(GX) + \tilde{g}(J_2(GX), \xi)\xi + \tilde{g}(J_2(GX), J_1\xi)J_1\xi - \tilde{g}(\tilde{X}, \xi)\xi \\ &\quad - \tilde{g}(\tilde{X}, J_1\xi)J_1\xi - g(X, J_2\xi)J_2\xi - g(X, J_3\xi)J_3\xi \\ (9) \quad &= G^2X - g(GX, J_2\xi)\xi + g(GX, J_3\xi)J_1\xi - \tilde{g}(\tilde{X}, \xi)\xi \\ &\quad - \tilde{g}(\tilde{X}, J_1\xi)J_1\xi - g(X, J_2\xi)J_2\xi - g(X, J_3\xi)J_3\xi. \end{aligned}$$

On the other hand, by the definition of  $J_2$ ,

$$(10) \quad J_2^2 \tilde{X} = -\tilde{X} = -X - \tilde{g}(\tilde{X}, \xi)\xi - \tilde{g}(\tilde{X}, J_1\xi)J_1\xi.$$

Comparing the tangent and normal components in (9) and (10), we get

$$(11) \quad G^2X - g(X, J_2\xi)J_2\xi - g(X, J_3\xi)J_3\xi = -X,$$

$$(12) \quad g(GX, J_2\xi) = g(GX, J_3\xi) = 0.$$

Now we define 1-forms  $u$  and  $v$ , and unit dual vector fields  $U$  and  $V$  with respect to  $g$  by

$$(13) \quad u(X) = g(X, J_2\xi), \quad v(X) = g(X, J_3\xi) = -(u \circ J)(X),$$

$$(14) \quad U = J_2\xi, \quad V = J_3\xi = JU.$$

By this definitions, (11) and (12) show respectively

$$(15) \quad G^2 = -id. + u \otimes U + v \otimes V, \quad u(GX) = v(GX) = 0.$$

Also, applying  $X = U$  and  $X = V$  to (13), we have  $GU = 0$  and  $GV = 0$  respectively. Since  $J_2$  is skew-symmetric with respect to  $\tilde{g}$ , i.e.  $\tilde{g}(J_2X, Y) = -\tilde{g}(X, J_2Y)$ , we have  $\tilde{g}(GX, Y) = -\tilde{g}(X, GY)$ . Similarly, applying  $J_3$  to (6), we have

$$\begin{aligned} J_3\tilde{X} &= HX + \tilde{g}(X, \xi)J_3\xi + \tilde{g}(J_3X, J_1\xi)J_1\xi + \tilde{g}(\tilde{X}, \xi)J_3\xi + g(\tilde{X}, J_1\xi)J_2\xi \\ &= HX + g(X, \xi)J_3\xi + g(X, J_2\xi)J_1\xi + \tilde{g}(\tilde{X}, \xi)J_3\xi + g(\tilde{X}, J_1\xi)J_2\xi, \end{aligned}$$

where  $HX$  is the component of  $J_3X$  tangent to  $TM$ , and some relations similar to (12), (13) and (14),

$$(16) \quad H^2 = -id. + u \otimes U + v \otimes V, \quad HU = HV = 0,$$

$$(17) \quad u \circ H = v \circ H = 0, \quad g(HX, Y) = -g(X, HY).$$

Now applying  $J_2$  to (15), we have

$$\begin{aligned} J_2J_3\tilde{X} &= J_2HX + \tilde{g}(\tilde{X}, \xi)J_1\xi - \tilde{g}(\tilde{X}, J_1\xi)\xi \\ &\quad - g(X, J_3\xi)J_2\xi + g(X, J_2\xi)J_3\xi \\ &= G(HX) - g(HX, J_2\xi)\xi + g(HX, J_3\xi)J_1\xi + \tilde{g}(\tilde{X}, \xi)J_1\xi \\ &\quad - \tilde{g}(\tilde{X}, J_1\xi)\xi - v(X)U + u(X)V \\ (18) \quad &= GHX - v(X)U + u(X)V + \tilde{g}(\tilde{X}, \xi)J_1\xi - \tilde{g}(\tilde{X}, J_1\xi)\xi. \end{aligned}$$

On the other hand,

$$J_2J_3\tilde{X} = J_1\tilde{X} = JX + \tilde{g}(\tilde{X}, \xi)J_1\xi - \tilde{g}(\tilde{X}, J_1\xi)\xi.$$

By comparing the tangent parts in (17) and (18), we get

$$(19) \quad GH = J + v \otimes U - u \otimes V.$$

Finally, applying  $G$  from left side to (14) and (21), we have respectively

$$\begin{aligned} G^2H &= G(GH) = GJ + v \otimes GU - u \otimes GV = GJ, \\ G^2H &= -H + (u \circ H) \otimes U + (v \circ H) \otimes V = -H, \end{aligned}$$

which show

$$(20) \quad GJ = -H.$$

From (13), (14), (15), (16), (17), (19) and (20), the structure  $(G, H, J, u, v, U, V, g)$  satisfies the definition of complex almost contact metric structure. Then we conclude our theorem.

**Theorem 3.3.** Let  $(\widetilde{M}, J_1, J_2, J_3, \tilde{g})$  be a hyperkähler manifold and  $M$  be a complex hypersurface of  $\widetilde{M}$ . The inclusion  $\iota : M \longrightarrow \widetilde{M}$  canonically induces a complex almost contact metric structure on  $M$ .

## 4 connections

This section is based on a paper by B. Smyth [8]. Let  $\tilde{\nabla}$  be the Levi-Civita connection with respect to  $\tilde{g}$ . For  $X, Y \in TM$ , we define a tensor field of type  $(1, 1)$   $A$  and a 1-form  $s$  by

$$\tilde{\nabla}_X \xi = -AX + s(X)J_1\xi.$$

Then we decompose  $\tilde{\nabla}_X Y$  to

$$\tilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)\xi + g(JAX, Y)J_1\xi,$$

where  $\nabla_X Y$  denotes the component of  $\tilde{\nabla}_X Y$  tangent to  $M$ . It is known that  $\nabla$  is the Levi-Civita connection with respect to  $g$ . Now we give expressions for the covariant derivatives of  $G$  and  $H$  on a complex almost contact metric structure on  $M$ .

**Theorem 4.1** Let  $(G, H, J, u, v, U, V, g)$  be a complex almost contact metric structure on a complex hypersurface  $M$ . Then the derivatives of  $G$  and  $H$  have the following forms:

$$\begin{aligned} (\nabla_X G)Y &= -u(Y)AX + v(Y)JAX + g(AX, Y)U - g(JAX, Y)V, \\ (\nabla_X H)Y &= -u(Y)JAX - v(Y)AX + g(AX, Y)V + g(JAX, Y)U. \end{aligned}$$

*Proof.* Since from (7),  $J_2Y = GY + \tilde{g}(J_2Y, \xi)\xi + \tilde{g}(J_2Y, J_1\xi)J_1\xi$ , we have

$$\begin{aligned}
\tilde{\nabla}_X(J_2Y) &= \tilde{\nabla}_X(GY) + \tilde{g}(J_2\tilde{\nabla}_X Y, \xi)\xi + \tilde{g}(J_2Y, \tilde{\nabla}_X \xi)\xi + \tilde{g}(J_2Y, \xi)\tilde{\nabla}_X \xi \\
&\quad + \tilde{g}(J_2\tilde{\nabla}_X Y, J_1\xi)J_1\xi + \tilde{g}(J_2Y, J_1\tilde{\nabla}_X \xi)J_1\xi + \tilde{g}(J_2Y, J_1\xi)J_1\tilde{\nabla}_X \xi \\
&= \nabla_X(GY) + g(AX, GY)\xi + g(JAX, GY)J_1\xi - \tilde{g}(\tilde{\nabla}_X Y, J_2\xi)\xi \\
&\quad + \tilde{g}(J_2Y, -AX + s(X)J_1\xi)\xi - \tilde{g}(Y, J_2\xi)(-AX + s(X)J_1\xi) \\
&\quad + \tilde{g}(\tilde{\nabla}_X Y, J_3\xi)J_1\xi + \tilde{g}(J_2Y, -JAX - s(X)\xi)J_1\xi \\
&\quad + \tilde{g}(Y, J_3\xi)(-JAX - s(X)J_1\xi) \\
&= \nabla_X(GY) + g(AX, GY)\xi + g(JAX, GY)J_1\xi - u(\nabla_X Y)\xi \\
&\quad - g(GY, AX)\xi + s(X)v(Y)\xi + u(Y)AX - s(X)u(Y)J_1\xi \\
&\quad + v(\nabla_X Y)J_1\xi - g(GY, JAX)J_1\xi + s(X)u(Y)J_1\xi \\
&\quad - v(Y)JAX - s(X)v(Y)\xi \\
(21) \quad &= (\nabla_X G)Y + G\nabla_X Y - u(\nabla_X Y)\xi + u(Y)AX + v(\nabla_X Y)J_1\xi \\
&\quad - v(Y)JAX.
\end{aligned}$$

On the other hand, since  $J_2$  is parallel to  $\tilde{\nabla}$ , we have

$$\begin{aligned}
\tilde{\nabla}_X(J_2Y) &= J_2\tilde{\nabla}_X Y \\
&= J_2(\nabla_X Y + g(AX, Y)\xi + g(JAX, Y)J_1\xi) \\
&= G\nabla_X Y + \tilde{g}(J_2\nabla_X Y, \xi)\xi + \tilde{g}(J_2\nabla_X Y, J_1\xi)J_1\xi \\
&\quad + g(AX, Y)\xi - g(JAX, Y)J_1\xi \\
(22) \quad &= G\nabla_X Y - u(\nabla_X Y)\xi + v(\nabla_X Y)J_1\xi + g(AX, Y)\xi - g(JAX, Y)J_1\xi
\end{aligned}$$

Comparing (21) and (22), we get

$$\begin{aligned}
(23) \quad (\nabla_X G)Y &= -u(Y)AX + v(Y)JAX \\
&\quad + g(AX, Y)U - g(JAX, Y)V.
\end{aligned}$$

Since  $J$  is parallel to  $\nabla$ , we have

$$J(\nabla_X G)Y = \nabla_X(JGY) - JG\nabla_X Y = (\nabla_X H)Y.$$

Then applying  $J$  to (23), we get

$$(\nabla_X H)Y = -u(Y)JAX - v(Y)AX + g(AX, Y)V + g(JAX, Y)U.$$

**Proposition 4.2** Let  $(G, H, J, u, v, U, V, g)$  be a complex almost contact metric structure on a complex hypersurface  $M$ . Then

$$s(X) = g(\nabla_X V, U) = -\sigma(X).$$

*Proof.* By Proposition 4.1, we have

$$\begin{aligned}
G(\nabla_X G)Y &= \nabla_X(G^2 Y) - G^2 \nabla_X Y - (\nabla_X G)(GY) \\
&= \nabla_X(-Y + u(Y)U + v(Y)V) + \nabla_X Y - u(\nabla_X Y)U - v(\nabla_X Y)V \\
&\quad - g(AX, GY)U + g(JAX, GY)V \\
&= -\nabla_X Y + g(\nabla_X Y, U)U + g(Y, \nabla_X U)U + g(Y, U)\nabla_X U \\
&\quad + g(\nabla_X Y, V)V + g(Y, \nabla_X V)V + g(Y, V)\nabla_X V + \nabla_X Y \\
&\quad - u(\nabla_X Y)U - v(\nabla_X Y)V - g(AX, GY)U + g(JAX, GY)V \\
(24) \quad &= g(Y, \nabla_X U)U + u(Y)\nabla_X U + g(\nabla_X V, Y)V + v(Y)\nabla_X V \\
&\quad + g(GAX, Y)U + g(HAX, Y)V.
\end{aligned}$$

On the other hand, by proposition 4.1, we have

$$(25) \quad G(\nabla_X G)Y = -u(Y)GAX - v(Y)HAX.$$

Comparing (24) and (25), we get

$$\begin{aligned}
(26) \quad &g(Y, \nabla_X U)U + u(Y)\nabla_X U + g(\nabla_X V, Y)V + v(Y)\nabla_X V \\
&+ g(GAX, Y)U + g(HAX, Y)V + u(Y)GAX - v(Y)HAX = 0.
\end{aligned}$$

Applying  $u$  to (26), we have  $\nabla_X U = -GAX - u(\nabla_X V)V$ , since for  $X$  and  $Y$ ,

$$\begin{aligned}
0 &= g(Y, \nabla_X U) + v(Y)u(\nabla_X V) + g(GAX, Y) \\
&= g(Y, \nabla_X U + u(\nabla_X V)V + GAX).
\end{aligned}$$

By this expression, we get

$$\begin{aligned}
(27) \quad \tilde{\nabla}_X U &= \nabla_X U + g(AX, U)\xi + g(JAX, U)J_1\xi \\
&= -GAX - u(\nabla_X V)V + u(AX)\xi - v(AX)J_1\xi.
\end{aligned}$$

On the other hand, since  $J_2$  is parallel to  $\tilde{\nabla}$ , we have

$$\begin{aligned}
(28) \quad \tilde{\nabla}_X U &= \tilde{\nabla}_X(J_2\xi) \\
&= J_2(-AX + s(X)J_1\xi) \\
&= -GAX - \tilde{g}(J_2AX, \xi)\xi + \tilde{g}(J_2AX, J_1\xi)J_1\xi - s(X)J_3\xi \\
&= -GAX + u(AX)\xi - v(AX)J_1\xi - s(X)V.
\end{aligned}$$

Comparing (27) and (28), we get the conclusion.

**Proposition 4.3** For any  $X \in TM$ ,  $\nabla_X G$  and  $\nabla_X H$  are skew-symmetric operators with respect to  $g$ .



*Proof.* By proposition 4.1, we get the following equality, which gives the conclusion.

$$\begin{aligned}
g((\nabla_X G)Y, Z) &= -u(Y)g(AX, Z) + v(Y)g(JAX, Z) \\
&\quad + g(AX, Y)u(Z) - g(JAX, Y)v(Z) \\
&= -g((\nabla_X G)Z, Y).
\end{aligned}$$

## References

- [1] D. Blair, *Riemannian Geometry of Contact and Symplectic Manifolds* second edition, Progress in Mathematics 203, Birkhäuser, 2010.
- [2] M. Djoric and M. Okumura, *CR Submanifolds of Complex Projective Space*, Developments in Mathematics 19, Springer, Berlin, 2009.
- [3] S. Ishihara and M. Konishi, *Complex almost contact manifolds*, Kōdai Math. J. **3** (1980), 385-396.
- [4] S. Ishihara and M. Konishi, *Complex almost contact structures in a complex contact manifold*, Kōdai Math. J. **5** (1982), 30-37.
- [5] S. Kobayashi, *Remarks on complex contact manifolds*, Proc. Amer. Math. Soc. **10** (1959), 164-167.
- [6] B. Korkmaz, *Normality of complex contact manifolds*, Rocky Mountain J. of Math. **30** (2000), 1343-1380.
- [7] A. Morimoto, *On normal almost contact structures*, J. Math. Soc. Japan., **15** (1963), 420-436.
- [8] B. Smyth, *Differential geometry of complex hypersurfaces*, Ann. of Math., **85**(1967), 246-266.
- [9] Y. Tashiro, *On contact structures of hypersurfaces in complex manifolds I*, Tôhoku Math. J., **15**(1963), 62-78.

Department of Natural Sciences  
National Institute of Technology, Ibaraki Collage  
886 Nakane, Hitachinaka, Ibaraki 312-8508 Japan  
Tel: +81-29-271-2865 e-mail: imadam@ge.ibaraki-ct.ac.jp